

**VARIABLE DISAGGREGATION IN NETWORK FLOW  
PROBLEMS WITH PIECEWISE LINEAR COSTS**

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September 2003

## **Abstract**

We study mixed-integer programming formulations, based upon variable disaggregation, for generic multi-commodity network flow problems with non-convex piecewise linear costs, a problem class that arises frequently in many application domains in telecommunications, transportation, and logistics. We present several structural results for these formulations and we analyze the results of extensive experiments on a large set of instances with various characteristics. In particular, we show that the linear programming relaxation of an extended disaggregated model approximates the objective function by its lower convex envelope in the space of commodity flows. Together, the theoretical and computational results allow us to make suggestions concerning which formulation might be the most appropriate, depending on the characteristics of the problem instances.

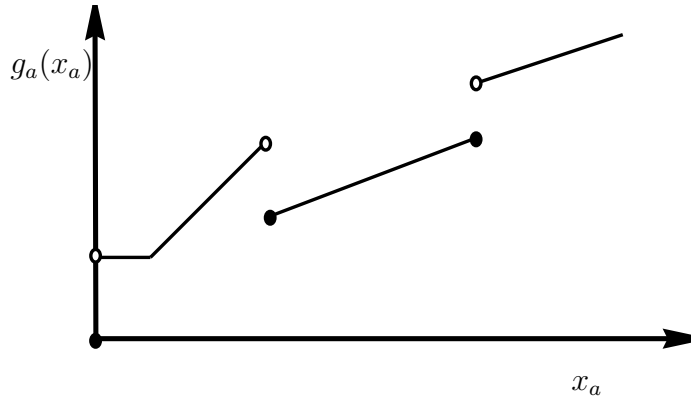


Figure 1: A Typical Piecewise Linear Cost Function

## 1 Introduction

Single and multi-commodity network flow problems with non-convex piecewise linear costs arise in many application areas, including telecommunications, transportation and logistics. We consider mixed-integer programming (MIP) formulations of a generic multi-commodity network flow problem with piecewise linear costs. The formulations we study are based on *variable disaggregation* techniques, which have been used over time to derive strong MIP formulations for variants of the fixed-charge network flow problem, a special case of our generic problem, but have not been studied extensively within the context of general non-convex piecewise linear cost functions.

Given a directed network  $G = (V, A)$ , with node set  $V$ , arc set  $A$ , supplies and demands of multiple commodities at the nodes, and arc capacities, we consider the problem of finding the minimum cost multi-commodity flow when the objective is the sum of  $|A|$  piecewise linear functions. As shown in Figure 1, we assume the cost  $g_a(x_a)$  on each arc  $a$  of the network is an arbitrary piecewise linear cost function of the arcs total flow,  $x_a$ , with each piece having nonnegative slope and intercept. The function need not be continuous and can have positive or negative jumps, though we assume that it is lower semi-continuous, that is,  $g_a(x_a) \leq \liminf_{x'_a \rightarrow x_a} g_a(x'_a)$  for any sequence  $x'_a$  that approaches  $x_a$ . Without loss of generality, we also assume, through a simple translation of the costs if necessary, that  $g_a(0) = 0$ .

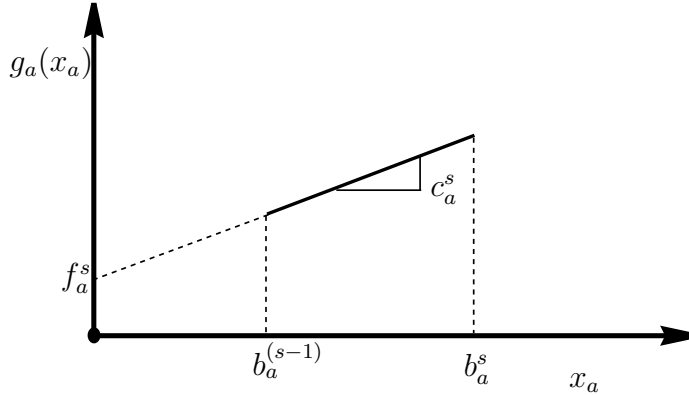


Figure 2: Notation for Each Segment

We can fully characterize a piecewise linear function by its segments. Figure 2 illustrates our notation. On each arc  $a$ , each segment  $s$  of the function has a nonnegative variable cost,  $c_a^s$  (the slope), a nonnegative fixed cost,  $f_a^s$  (the intercept), and upper and lower flow bounds,  $b_a^{s-1}$  and  $b_a^s$ . Since the total flow on each arc can always be bounded from above by either the arc capacity or the total demand flowing through the network, we assume each arc  $a$  has a finite number of segments, which we represent by the set  $S_a = \{1, 2, \dots, |S_a|\}$ .

As additional notation, we let  $K$  denote the set of commodities,  $N$  the  $|V| \times |A|$  node-arc incidence matrix, and  $d^k$  a vector of size  $|V|$  representing supplies and demands for commodity  $k$ . We can decompose the flow  $x_a$  on each arc  $a$  in two ways, by commodity or by segment, with  $x_a^k$  and  $x_a^s$  representing the flow of commodity  $k$  or the flow on segment  $s$ , respectively. More specifically,  $x_a^s$  is the total flow on arc  $a$  if that flow lies in segment  $s$ . Therefore, when a flow value of  $\hat{x}_a$  lies in segment  $\hat{s}$ ,  $x_a^{\hat{s}} = \hat{x}_a$  and  $x_a^s = 0$  for all segments  $s \neq \hat{s}$ . We also define binary variables,  $y_a^s$ , with  $y_a^s = 1$  if segment  $s$  contains a nonzero flow, and  $y_a^s = 0$  otherwise. With this notation, we can express the piecewise linear cost network flow problem (PLCNF) as the following MIP formulation:

$$\min \sum_{a \in A} \sum_{s \in S_a} c_a^s x_a^s + f_a^s y_a^s, \quad (1)$$

subject to:

$$Nx^k = d^k, \quad k \in K, \quad (2)$$

$$x_a = \sum_{k \in K} x_a^k, \quad a \in A, \quad (3)$$

$$x_a = \sum_{s \in S_a} x_a^s, \quad a \in A, \quad (4)$$

$$b_a^{s-1} y_a^s \leq x_a^s \leq b_a^s y_a^s, \quad a \in A, s \in S_a, \quad (5)$$

$$\sum_{s \in S_a} y_a^s \leq 1, \quad a \in A, \quad (6)$$

$$x_a^k \geq 0, \quad a \in A, k \in K, \quad (7)$$

$$y_a^s \in \{0, 1\}, \quad a \in A, s \in S_a. \quad (8)$$

By definition, we set  $b_a^0 = 0$  for every arc  $a$ . We refer to this formulation as the *basic* model. Constraints (2) are the flow balance constraints typical in a multi-commodity network flow formulation. For each node  $i$  and each commodity  $k$ ,  $d_i^k > 0$  denotes an origin node with supply  $d_i^k$ ,  $d_i^k < 0$  denotes a destination node with demand  $-d_i^k$ , and  $d_i^k = 0$  denotes a transshipment node. Constraints (3) and (4) define the flow by commodity and by segment, respectively. The *basic forcing constraints*, (5), state that if  $y_a^s = 0$ , then segment  $s$  has no flow, i.e.,  $x_a^s = 0$ . If  $y_a^s = 1$ , then the total flow on arc  $a$  must lie between the lower and upper bounds of that segment, i.e.,  $b_a^{s-1} \leq x_a^s \leq b_a^s$ . Inequality (6) assures that we choose at most one  $y_a^s$  to be positive on each arc  $a$  (when  $\sum_{s \in S_a} y_a^s = 0$ , arc  $a$  has zero flow and zero flow cost). Notice that we do not explicitly consider arc capacities, as we can model them through the definition of the cost function, by setting the bound  $b_a^{|S_a|}$  on the last segment equal to the capacity of arc  $a$ .

Since the fixed-charge network flow version of the model PLCNF is known to be NP-hard, PLCNF is NP-hard as well, though several well-known special cases, most notably when the cost function is convex, are known to be polynomially solvable. To solve the PLCNF, we would like to understand, and improve, the linear programming (LP) relaxations of the basic model. Croxton, Gendron, and Magnanti [14] prove that the LP relaxation of the basic model (as well as two alternative textbook formulations) approximates the piecewise linear cost function with its lower convex envelope. Figure 3 illustrates the lower convex envelope for the *concave case*, i.e., when the cost function is concave (thus satisfying  $f_a^{s-1} \leq f_a^s$  and  $c_a^{s-1} \geq c_a^s, \forall s \in S_a$ ). For this problem class, the

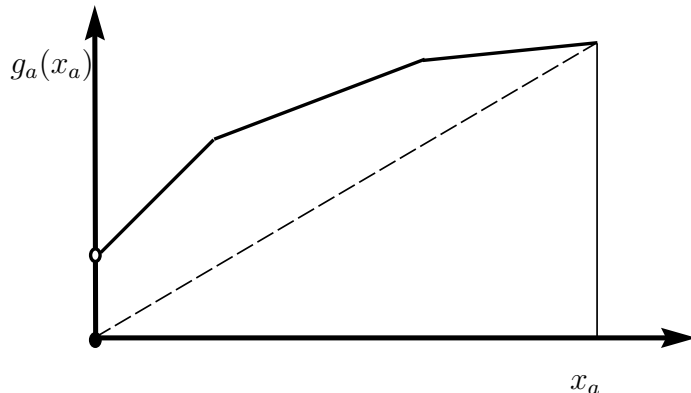


Figure 3: Lower Convex Envelope of a Concave Function

lower convex envelope is simply the line joining the origin to the point of maximum flow,  $(x_a^{|S_a|}, g_a(x_a^{|S_a|}))$ .

In general, the approximation provided by the lower convex envelope is rather poor, as evidenced by the computational experiments reported in Section 6. To improve upon it, a classical MIP approach would add valid inequalities that tighten the LP relaxation. The basic model lends itself naturally to two such simple classes of valid inequalities, which are similar to the basic forcing constraints (5). These classes of inequalities use the fact that the flow of commodity  $k$  on each arc  $a$  can be bounded from above by some constant  $M_a^k$ , usually set to the total supply for that commodity. In the particular case when each commodity has a single origin  $O(k)$  and a single destination  $D(k)$ , which we call the *single O-D* case, the bound  $M_a^k$  simply equals  $d_{O(k)}^k$ , the supply for commodity  $k$ .

The first class of valid inequalities, called the *strong forcing constraints*,

$$x_a^k \leq M_a^k \sum_{s \in S_a} y_a^s, \quad a \in A, k \in K, \quad (9)$$

state that when arc  $a$  has no flow (i.e.,  $\sum_{s \in S_a} y_a^s = 0$ ), the flow of each individual commodity is zero, and when the arc carries flow (i.e.,  $\sum_{s \in S_a} y_a^s = 1$ ), the flow of commodity  $k$  is bounded by  $M_a^k$ . In the remainder of this paper, we use the term *strong model* to designate the PLCNF with the addition of the strong forcing constraints.

As compared with constraints (5), the strong forcing constraints (9) disaggregate

commodities, but aggregate segments. If we could identify valid inequalities that disaggregate both commodities and segments, we would obtain stronger valid inequalities. Suppose we define additional variables  $x_a^{ks}$  that equal the flow of commodity  $k$  on arc  $a$  if the total flow on the arc lies in segment  $s$ , and equal zero otherwise. These new, *extended variables*, are related to the previous ones via the simple equations:  $x_a^s = \sum_{k \in K} x_a^{ks}$  and  $x_a^k = \sum_{s \in S_a} x_a^{ks}$ . Using them, we can define valid *extended forcing constraints*:

$$x_a^{ks} \leq M_a^k y_a^s, \quad a \in A, k \in K, s \in S_a, \quad (10)$$

that are obviously stronger than the strong forcing constraints, since we obtain the latter by aggregating over the segments. We refer to the model obtained by adding the non-negative variables  $x_a^{ks}$ , the extended forcing constraints, and the definitional equations  $x_a^s = \sum_{k \in K} x_a^{ks}$  and  $x_a^k = \sum_{s \in S_a} x_a^{ks}$  to the PLCNF as the *extended model*.

Note that when the cost function has a single segment as in the fixed-charge network design problem, the strong and extended forcing constraints are the same. Our choice of terminology for the strong forcing constraints is consistent with this special case. In this special case, if  $b_a^1 = \sum_k M_a^k$ , the so-called weak forcing constraint  $x_a^1 \leq b_a^1 y_a^1$  is an aggregation and so weaker than the strong forcing constraints. One question naturally arises: under what conditions does the extended model improve upon the strong one? We might also ask a similar question about the improvement achieved by using the LP relaxation of either model in place of that of the basic formulation. The main objective of this paper is to qualify, through theoretical results, as well as computational experiments, the improvement obtained by the addition of either the strong or the extended forcing constraints to the basic PLCNF model.

The paper is organized as follows. In Section 2, we review the relevant literature. In Section 3, we provide bounds on the improvement obtained by the addition of the strong forcing constraints, both in the concave case and in the general case. In Section 4, we analyze the improvement provided by the extended forcing constraints. More specifically, we generalize a result of Croxton, Gendron, and Magnanti [14] by showing that the LP relaxation of the extended model approximates the piecewise linear cost function by its lower convex envelope represented in the space of commodity flows, rather than in the

space of aggregated flows, as does the basic model. We also specify conditions guaranteeing the integrality of the LP relaxation of the extended formulation. Section 5 provides a geometrical interpretation of the results from Sections 3 and 4. In Section 6, we present and analyze the results of extensive computational experiments on a large set of instances with various characteristics, and use them to confirm our theoretical investigations. In particular, we examine different cost structures, concave and non-concave, for situations when each commodity has one origin, but multiple destinations, studying the impact of reformulating these instances as single O-D networks, a classic variable disaggregation technique in the context of multi-commodity network flow problems. Together, the theoretical and computational results allow us to suggest when each of the formulations might be the most appropriate. Through this analysis we aim, therefore, to contribute to both theory and practice. Finally, in Section 7, we summarize our results.

## 2 Prior Literature

Although the literature on network flows is extensive, research concerning problems with piecewise linear cost functions is limited, except for the concave and fixed-charge cases. Balakrishnan and Graves [3] develop a Lagrangian-based algorithm for the uncapacitated network flow problem with piecewise linear concave costs. Although their approach is generic and applies to the non-concave case, we are not aware of computational experiments reporting on the performance of this algorithm on networks with non-concave costs. Cominetti and Ortega [11] use a branch-and-bound algorithm to solve the capacitated network flow problem with piecewise linear concave costs. They exploit the concavity of the cost function and employ sensitivity analysis to improve the approximations and obtain better lower bounds. Motivated by a logistics application, Chan, Muriel, and Simchi-Levi [10] examine the multi-commodity version of the same problem. They derive structural results of a set-partitioning formulation, and then use this insight to develop a linear programming based heuristic. Aghezzaf and Wolsey [1] study piecewise linear concave costs in a tree partitioning problem.

Other relevant work on network flows includes research by Popken [33] who develops



an algorithm for the multi-commodity flow problem with continuously differentiable non-concave costs. Kim and Pardalos [28] propose an approximation algorithm for the non-convex piecewise linear network flow problem, which iteratively solves a series of linear network flow problems with dynamically adapted costs and restricted domains. Holmberg [24] develops an algorithm for the facility location problem with staircase costs based upon convex linearization and Bender’s decomposition. Holmberg and Ling [26] use a Lagrangian heuristic for the same problem.

The class of problems in network design is also related to our work, since the cost functions for these problems are usually discontinuous. Magnanti and Wong [31] and Minoux [32] survey network design issues in transportation planning and telecommunications, respectively. Balakrishnan, Magnanti, and Mirchandani [4] provide an annotated bibliography concerning general network design issues. Most of the research in the area of network design has focused on the fixed-charge case [5, 12, 21, 25, 27], or on variants of the network loading problem, which arise in telecommunications applications [2, 6, 7, 8, 17, 18, 23, 30].

Within the context of multi-commodity network flow problems, many researchers have addressed the idea of variable disaggregation. Rardin and Wolsey [34] study the MIP formulations of multi-commodity uncapacitated fixed charge network flow problems, when each commodity has a single origin and multiple destinations. They develop inequalities that fully describe the projection onto the original variable space of a disaggregated formulation obtained by representing each commodity with a single origin and a single destination. Bienstock and Günlük [9] study a variant of the multi-commodity network flow problem with the objective of minimizing the maximum load on any edge. They discuss the possibility of using a variable disaggregation by commodity, but found that for their problem the improvement in the lower bounds generated by the LP relaxation was not significant enough to warrant the increased computational power required.

Multi-commodity network flow problems with piecewise linear costs commonly arise in transportation and logistics applications. For example, Croxton, Gendron and Magnanti [15] exploit the disaggregation techniques discussed in the present paper to solve large-

scale instances of a logistics application called the merge-in-transit problem.

### 3 The Strong Model

In terms of the strength of their LP relaxations, the strong model sits between the basic and extended models. We already noted that it is not, in general, as tight (as measured by the objective value of its linear programming relaxation) as the extended formulation. However, it is in general tighter than the basic model, and it is smaller than the extended model, with fewer variables and constraints. The following result provides an upper bound on the improvement obtained by adding the strong forcing constraints (9) to the basic model.

**Proposition 1** *Let  $(\hat{x}, \hat{y})$  be an optimal solution to the LP relaxation of the basic model, with a value of  $\widehat{Z}^l$ , and let  $A^+ = \{a \in A : \hat{x}_a > 0\}$  be the set of arcs with positive flow in this solution. The lower bound provided by the LP relaxation of the strong model improves upon  $\widehat{Z}^l$  by at most  $\sum_{a \in A^+} f_a^1(1 - \sum_{s \in S_a} \hat{y}_a^s)$ .*

**Proof:** If  $(\hat{x}, \hat{y})$  satisfies the inequalities (9), then this solution satisfies all the constraints of the strong formulation and must be optimal. Otherwise, only arcs with positive flow, i.e., arcs in  $A^+$ , can violate the constraints (9). For these arcs, it suffices to change one or more of the values of the  $y$  variables so that they satisfy these constraints. In the worst case (e.g., when  $x_a^k = M_a^k$ ), we can increase  $y_a^1$  for each arc in  $A^+$  until  $\sum_{s \in S_a} y_a^s = 1$  without violating any other constraints because the model imposes no upper bound on  $y_a^1$  (since  $b_a^0 = 0$ ). We now have constructed a modified solution,  $(x, y)$ , that is feasible for the strong formulation and that satisfies  $y_a^1 = 1 - \sum_{s \neq 1} \hat{y}_a^s$ . The cost of the original and new solutions are related by the following:  $cost(x, y) = cost(\hat{x}, \hat{y}) + \sum_{a \in A^+} f_a^1(y_a^1 - \hat{y}_a^1) = cost(\hat{x}, \hat{y}) + \sum_{a \in A^+} f_a^1(1 - \sum_{s \in S_a} \hat{y}_a^s)$ . Since  $\widehat{Z}^l = cost(\hat{x}, \hat{y})$  and the optimal cost of the strong formulation is at most  $cost(x, y)$ ,  $\sum_{a \in A^+} f_a^1(1 - \sum_{s \in S_a} \hat{y}_a^s)$  is an upper bound on the improvement obtained by the strong formulation. ■

In the concave case, we can improve the upper bound provided by Proposition 1.

**Proposition 2** *Let  $(\hat{x}, \hat{y})$  be an optimal solution to the LP relaxation of the basic model, with a value of  $\widehat{Z}^l$ , and define the set  $A^+ = \{a \in A : \hat{x}_a > 0\}$ . In the concave case, the lower bound provided by the LP relaxation of the strong model improves upon  $\widehat{Z}^l$  by at most  $\sum_{a \in A^+} f_a^1 \max\{0, \max_{k \in K} \{\frac{\hat{x}_a^k}{M_a^k}\} - \frac{\hat{x}_a}{b_a^{|S_a|}}\}$ .*

**Proof:** In the concave case, we know that for each arc with positive flow, the solution to the LP relaxation of the basic formulation is  $\hat{x}_a^{|S_a|} = \hat{x}_a$  and  $\hat{y}_a^{|S_a|} = \frac{\hat{x}_a}{b_a^{|S_a|}}$ . Since  $f_a^1 = \min_{s \in S_a} \{f_a^s\}$ , it is cheaper to increase  $y_a^1$  than any of the other  $y_a$ 's. Therefore, by letting  $y_a^1 = \max_{k \in K} \frac{\hat{x}_a^k}{M_a^k}$  and  $y_a^s = 0$ ,  $s \neq 1$ , we satisfy the inequality (9) and have found a solution to the strong formulation when we maintain the same arc flows. This solution increases the objective value by  $\sum_{a \in A^+} f_a^1 \max\{0, \max_{k \in K} \{\frac{\hat{x}_a^k}{M_a^k} - \frac{\hat{x}_a}{b_a^{|S_a|}}\}$ . Since we are minimizing, the LP optimal value, and therefore the lower bound, increases by at most this amount. ■

Propositions 1 and 2 not only characterize upper bounds on the improvement obtained by adding the strong valid inequalities (9), but they also identify an important situation when there is no improvement.

**Corollary 3** *When the cost functions have no initial fixed cost (i.e.,  $f_a^1 = 0$ , for all arcs  $a \in A$ ), the strong formulation does not improve upon the basic formulation.*

## 4 The Extended Model

For the extended model, given by the formulation (1)-(8), the extended forcing constraints (10), and the definitional equations  $x_a^s = \sum_{k \in K} x_a^{ks}$  and  $x_a^k = \sum_{s \in S_a} x_a^{ks}$  for the extended variables, we examine two structural results. The first result, the *lower convex envelope property*, states that the LP relaxation of the extended formulation approximates the cost function with its lower convex envelope represented in the space of commodity flows. The second result, the *single-path property*, specifies that the LP optimal solution is integral whenever it satisfies the demand for each commodity through a single path.

## 4.1 The Lower Convex Envelope Property

Suppose we consider the cost on any arc  $a$ , represented as follows, as a function of the  $|K|$  variables,  $x_a^1, x_a^2, \dots, x_a^{|K|}$ :

$$g_a(x_a^1, x_a^2, \dots, x_a^{|K|}) = \min \sum_{s \in S_a} c_a^s x_a^s + f_a^s y_a^s, \quad (11)$$

subject to:

$$x_a^k = \sum_{s \in S_a} x_a^{ks}, \quad a \in A, k \in K, \quad (12)$$

$$x_a^s = \sum_{k \in K} x_a^{ks}, \quad a \in A, s \in S_a, \quad (13)$$

$$b_a^{s-1} y_a^s \leq x_a^s \leq b_a^s y_a^s, \quad a \in A, s \in S_a, \quad (14)$$

$$0 \leq x_a^{ks} \leq M_a^k y_a^s, \quad a \in A, k \in K, s \in S_a, \quad (15)$$

$$\sum_{s \in S_a} y_a^s \leq 1, \quad a \in A, \quad (16)$$

$$y_a^s \in \{0, 1\}, \quad a \in A, s \in S_a. \quad (17)$$

In this representation of the costs as an optimization model, the values of the parameters  $x_a^k$  are fixed, the variables being  $x_a^{ks}$ ,  $x_a^s$  and  $y_a^s$ .

Using this complex representation of the cost functions  $g_a$ , we can formulate PLCNF as a non-convex nonlinear model, NON-PLCNF, with simple network flow constraints:

$$\min \sum_{a \in A} g_a(x_a^1, x_a^2, \dots, x_a^{|K|}), \quad (18)$$

subject to:

$$Nx^k = d^k, \quad k \in K, \quad (19)$$

$$\sum_{k \in K} x_a^k \leq u_a, \quad a \in A, \quad (20)$$

$$0 \leq x_a^k \leq M_a^k, \quad a \in A, k \in K. \quad (21)$$

In this formulation,  $u_a = b_a^{|S_a|}$  is the capacity of arc  $a$ .

To prove the lower convex envelope property, we use an argument based upon Lagrangian duality and a theorem of Falk [16] that relates the Lagrangian dual to the lower

convex envelope of the cost function (see the Appendix for an alternative proof, based upon a characterization of the extreme points of the polyhedron defined by the LP relaxation of the extended formulation). To define the Lagrangian dual with respect to NON-PLCNF, we associate Lagrangian multipliers  $\pi^k$  and  $\alpha_a \geq 0$  with the constraints (19) and (20), let  $g_a^{(\pi, \alpha)}(x_a^1, x_a^2, \dots, x_a^{|K|}) = g_a(x_a^1, x_a^2, \dots, x_a^{|K|}) + \sum_{k \in K} (\pi_i^k - \pi_j^k + \alpha_a) x_a^k$  for each arc  $a = (i, j)$ , and form a Lagrangian subproblem,  $LS(\pi, \alpha)$ , written as:

$$Z_{LS}(\pi, \alpha) = \min \sum_{a \in A} g_a^{(\pi, \alpha)}(x_a^1, x_a^2, \dots, x_a^{|K|}), \quad (22)$$

subject to the bounding constraints (21). The Lagrangian dual,  $LD$ , is:

$$Z_{LD} = \max_{\pi; \alpha \geq 0} Z_{LS}(\pi, \alpha) - \sum_{k \in K} d^k \pi^k - \sum_{a \in A} u_a \alpha_a. \quad (23)$$

By a direct adaptation of Falk's theorem to our model, we obtain the following result.

**Theorem 4** *Let  $(\pi, \alpha)^*$  be an optimal solution to  $LD$  and  $x^*$  be an optimal solution to the corresponding Lagrangian subproblem,  $LS((\pi, \alpha)^*)$ . Then,  $x^*$  minimizes the lower convex envelope of the cost function over the polyhedron defined by constraints (19) to (21).*

We will use this result to prove the following lower convex envelope property:

**Theorem 5** *The LP relaxation of the extended formulation approximates the cost function  $\sum_{a \in A} g_a(x_a^1, x_a^2, \dots, x_a^{|K|})$  with its lower convex envelope, in the sense that its optimal value equals the minimum of the lower convex envelope of this function over the feasible region.*

**Proof:** Substituting for the function  $g_a(x_a^1, x_a^2, \dots, x_a^{|K|})$ , we can rewrite the Lagrangian subproblem  $LS(\pi, \alpha)$  as the following equivalent MIP model with the added binary variables  $y_a^s$  and the auxiliary variables  $x_a^{ks}$ :

$$Z_{LS}(\pi, \alpha) = \min \sum_{a \in A} \sum_{s \in S_a} \left\{ \sum_{k \in K} c_a^{ks}(\pi, \alpha) x_a^{ks} \right\} + f_a^s y_a^s, \quad (24)$$

subject to the constraints (6), (8), and (12)-(17) with  $c_a^{ks}(\pi, \alpha) = c_a^s + \pi_i^k - \pi_j^k + \alpha_a$  for  $a = (i, j)$ . Note that because of the extended forcing inequalities (10), the constraints (21) are now redundant and so we can remove them. Given that the capacity constraints (20) are implied by the basic forcing constraints (14), we know that  $LD$  has an optimal solution with  $\alpha = 0$ . Suppose we form a Lagrangian dual  $LD^E$  from the extended formulation by relaxing the flow conservation constraints (2).  $LD^E$  is defined as  $Z_{LD^E} = \max_{\pi} Z_{LS^E}(\pi) - \sum_{k \in K} d^k \pi^k$ , and the Lagrangian subproblem,  $LS^E(\pi)$ , is equivalent to  $LS(\pi, 0)$ . Consequently,  $LD^E$  is equivalent to  $LD$ , which implies, from Falk's theorem, that  $LD^E$  approximates the cost function by its lower convex envelope. We will now show that  $LD^E$  is equivalent to the LP relaxation of the extended formulation, which will conclude our argument.

To establish this equivalence, by the well known integrality property ([22]), it suffices to prove that the LP relaxation of  $LS^E(\pi)$  always has an integral optimal solution. For each arc  $a$ , let  $x_a = (x_a^{ks})$  and  $y_a = (y_a^s)$  denote vectors of commodity-segment flows and binary segment choice variables. We first note that  $LS^E(\pi)$  decomposes into  $|A|$  subproblems, one for each arc  $a$ , for which we define an associated polyhedron:  $P_a = \{(x_a, y_a) : b_a^{s-1} y_a^s \leq \sum_{k \in K} x_a^{ks} \leq b_a^s y_a^s, \forall s \in S_a; x_a^{ks} \leq M_a^k y_a^s, \forall s \in S_a, k \in K; \sum_{s \in S_a} y_a^s \leq 1; y_a^s \geq 0, \forall s \in S_a; x_a^{ks} \geq 0, \forall s \in S_a, k \in K\}$ . To obtain the desired result, we now show that for every extreme point of  $P_a$ ,  $y_a^s \in \{0, 1\}$  for all  $s \in S_a$ . If not, then let  $(\hat{x}, \hat{y})$  be an extreme point of  $P_a$  with at least one fractional component. Without loss of generality, we assume that  $0 < \hat{y}_a^t < 1$ , for  $t = 1, 2, \dots, r$ . Define the following  $r + 1$  points in  $P_a$ :  $(x(0), y(0)) = (0, 0)$  and  $(x(t), y(t)), 1 \leq t \leq r$ , with  $x_a^{kt}(t) = \frac{\hat{x}_a^{kt}}{\hat{y}_a^t}$ ,  $x_a^{ks}(t) = 0$ ,  $s \neq t$ ,  $y_a^t(t) = 1$  and  $y_a^s = 0$ ,  $s \neq t$ . Then  $(\hat{x}, \hat{y}) = (1 - \sum_{t=1}^r \hat{y}_a^t)(x(0), y(0)) + \sum_{t=1}^r \hat{y}_a^t (x(t), y(t))$ , is a representation of  $(\hat{x}, \hat{y})$  as a convex combination of  $r + 1$  distinct points in  $P_a$ , contradicting the hypothesis that it is an extreme point of  $P_a$ . ■

In Section 5, we give a geometric illustration of this lower convex envelope result for a two-commodity example on a single arc.

## 4.2 The Single-Path Property

Frequently, the LP relaxation of the extended formulation for uncapacitated single O-D networks with concave cost functions provides integral solutions (recall that in a single O-D network, each commodity is defined by a single origin and a single destination). The reason for this behavior is that in the LP optimal solution for such instances, each commodity tends to flow through a single path. We prove that this property implies integral LP solutions to the extended single O-D formulation.

**Proposition 6** *The LP relaxation of the extended formulation of single origin, single destination (single O-D) models has an integral optimal solution whenever in the LP solution, each commodity flows on a single path through the network.*

**Proof:** Let  $D^k$  denote the supply for commodity  $k$ . In the single O-D case,  $M_a^k = D^k$  and so the extended forcing constraints become:

$$x_a^{ks} \leq D^k y_a^s \quad a \in A, k \in K, s \in S_a. \quad (25)$$

Consider an arc  $a$  of the network on which a nonempty subset  $K'$  of the commodities flows. Since each commodity flows on a single path, for each  $k \in K'$ ,  $\sum_{s \in S_a} x_a^{ks} = D^k$ .

We now note three basic results, which we then use to demonstrate the existence of an integral LP optimal solution.

1. For each  $k \in K'$  and  $s \in S_a$ ,  $y_a^s \geq \frac{x_a^{ks}}{D^k} \implies \sum_{s \in S_a} y_a^s \geq \frac{\sum_{s \in S_a} x_a^{ks}}{D^k} = 1$  (since  $\sum_{s \in S_a} x_a^{ks} = D^k$ ). But  $\sum_{s \in S_a} y_a^s \leq 1 \implies \sum_{s \in S_a} y_a^s = 1$ .
2. The fact that  $\sum_{s \in S_a} y_a^s = 1$  and the fact that, for each  $k \in K'$ ,  $\sum_{s \in S_a} \frac{x_a^{ks}}{D^k} = 1 \implies \sum_{s \in S_a} (y_a^s - \frac{x_a^{ks}}{D^k}) = 0$ , for each  $k \in K'$ . But, for each  $k \in K'$  and  $s \in S_a$ ,  $y_a^s \geq \frac{x_a^{ks}}{D^k} \implies x_a^{ks} = y_a^s D^k$ .
3. Suppose that  $b_a^{\hat{s}-1} < \sum_{k \in K'} D^k < b_a^{\hat{t}}$ . Then  $y_a^s = 0$  for all  $s < \hat{s}$  and  $s > \hat{t}$ .

For any  $s < \hat{s}$ ,  $\sum_{k \in K'} D^k > b_a^s$ . Assume  $y_a^s > 0$ . Then,  $b_a^s y_a^s \geq \sum_{k \in K'} x_a^{ks} = y_a^s \sum_{k \in K'} D^k > y_a^s b_a^s \implies \text{contradiction} \implies y_a^s = 0, \forall s < \hat{s}$ .

For any  $s > \hat{t}$ ,  $\sum_{k \in K'} D^k < b_a^{s-1}$ . Assume  $y_a^s > 0$ . Then,  $b_a^{s-1} y_a^s \leq \sum_{k \in K'} x_a^{ks} = y_a^s \sum_{k \in K'} D^k < y_a^s b_a^{s-1} \implies \text{contradiction} \implies y_a^s = 0, \forall s > \hat{t}$ .

We now distinguish two cases that exhaust all possibilities:

1. For some  $\hat{s}$ ,  $b_a^{\hat{s}-1} < \sum_{k \in K'} D^k < b_a^{\hat{s}}$ . Our prior observations imply that  $y_a^{\hat{s}} = 1$  since  $\sum_{s \in S_a} y_a^s = 1$ , and  $y_a^s = 0$  for all  $s < \hat{s}$  or  $s > \hat{s}$ . Therefore,  $y_a^s \in \{0, 1\}, \forall s \in S_a$ .
2.  $\sum_{k \in K'} D^k = b_a^{\hat{s}}$  for some  $\hat{s}$ .

By our prior observations only  $y_a^{\hat{s}} > 0$  or  $y_a^{\hat{s}+1} > 0$  are possibilities. Substituting for  $x_a^s = \sum_{k \in K'} x_a^{ks} = y_a^s \sum_{k \in K'} D^k$  in the objective shows that the contribution of arc  $a$  to the cost is:  $F_a^{\hat{s}} y_a^{\hat{s}} + F_a^{\hat{s}+1} y_a^{\hat{s}+1}$ , with  $F_a^s = (c_a^s \sum_{k \in K'} D^k + f_a^s)$ . Therefore, the LP has an integral solution given by:  $y_a^{\hat{s}+1} = 1$  and  $y_a^s = 0, \forall s \neq \hat{s} + 1$ , if  $F_a^{\hat{s}+1} \leq F_a^{\hat{s}}$ , or  $y_a^{\hat{s}} = 1$  and  $y_a^s = 0, \forall s \neq \hat{s}$ , otherwise.

■

The LP solution to the extended single O-D formulation does not always satisfy this single-path property, even for uncapacitated concave cost instances. Indeed, Krarup and Pruzan [29] provide an instance of the uncapacitated facility location problem, which is a special case of the more general network flow problem with concave costs, with a fractional single O-D extended formulation. Nonetheless, Proposition 6 demonstrates why the solution to the LP relaxation of the single O-D extended formulation is often integral, and suggests that disaggregating single origin or single destination problems into single O-D problems can be quite effective for uncapacitated concave cost instances. For example, Croxton [13] reports results comparing the solution to the LP relaxation single O-D extended formulation with the solution provided by the B&B algorithm of Cominetti and Ortega [11] on an uncapacitated network with concave costs provided by these authors (this network has 100 nodes, 195 arcs, and 70 commodities, and each commodity has multiple origins and a single destination). Computational experiments show that the LP relaxation of a single O-D extended model provides integral solutions for all test problems. Moreover, solving the LP relaxation with a state-of-the-art LP software package proved



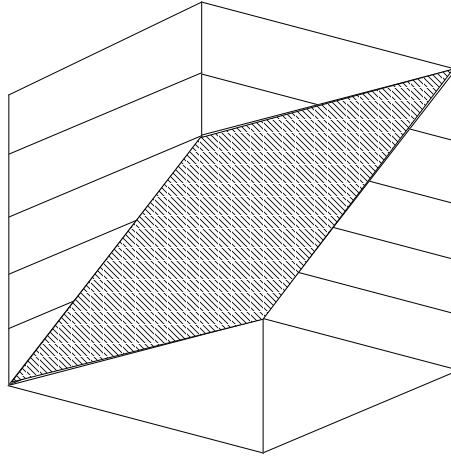
to be significantly faster than the implementation of the specialized algorithm developed by Cominetti and Ortega.

## 5 Geometric Interpretation

To help understand the implications of the valid inequalities, we examine a two-commodity example with concave costs on a single arc. Recall that, as shown by the dotted line in Figure 3, the approximation given by the LP relaxation of the basic formulation is the line segment from the origin to the point of maximum flow, as this is the lower convex envelope of  $g_a(x_a)$ .

When we disaggregate the total flow into two commodities, we can draw the cost function  $g_a(x_a^1, x_a^2)$  in three dimensions. Figure 4 illustrates the difference between the basic and the extended formulations for this example. The basic approximation, that is, the approximation of the cost function provided by the LP relaxation of the basic formulation, is a single plane that lies below the cost function, i.e., the dotted line expanded into three dimensions. The basic approximation and the actual cost function are equal at the origin and at the point of maximum flow, where each commodity is at its maximum, but at no other points. Figure 3 shows this behavior. On this plane we can draw a line, which we call the *diagonal*, from the origin to the point of maximum flow,  $b_a^{|S_a|}$ . The approximation of the cost function provided by the LP relaxation of the extended formulation, called the extended approximation, is formed by two planes that intersect along the diagonal. Each of these planes equals the actual cost function at three points: the origin, the point of maximum flow,  $b_a^{|S_a|}$ , and one of the two points where one commodity has zero flow and the other is at its maximum,  $M_a^k$  (called *corner points*). Along the diagonal, the basic and extended approximations are equal, but at all other points, the extended approximation improves upon the basic one. Geometrically, when we disaggregate the variables and use the extended formulation, we fold the plane formed by the basic approximation about the diagonal and lift each of the two resulting planes until their corners meet the actual cost function. The extent to which this folding lifts

Basic Approximation



Extended Approximation

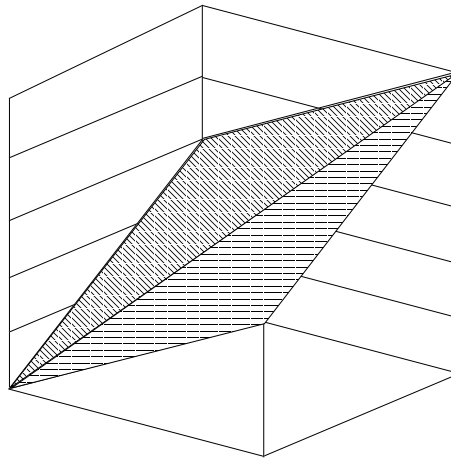


Figure 4: Comparing the Basic and Extended Approximations

the planes is a measure of the improvement achieved by using the extended formulation.

A similar analysis can help us interpret the strong formulation geometrically. We note that along the diagonal,  $\max\{0, \max_k \{\frac{\hat{x}_a^k}{M_a^k}\} - \frac{\hat{x}_a}{b_a^{|S_a|}}\} = 0$  and therefore as a result of Proposition 2, the basic and the strong LP solutions have the same cost. Elsewhere, in the strong formulation,  $y_a^1$  increases linearly along each dimension as  $\hat{x}_a^k$  increases. The result is a function that is folded along the diagonal much like that of the extended formulation, as shown in Figure 4, but with a fold that is not as high. Consequently, the strong formulation attains only a portion of the improvement achieved by the extended formulation.

For the extended formulation, we can generalize these results for the  $|K|$ -commodity case. In fact, Chan, Muriel and Simchi-Levi [10] provide closed form expressions for these planes in the concave case. If we manipulate their expressions, we can provide a geometric interpretation. To determine the cost of a flow vector  $(x_a^1, x_a^2, \dots, x_a^{|K|})$  in the extended formulation, we first order the commodities so that  $\frac{x_a^1}{M_a^1} \geq \frac{x_a^2}{M_a^2} \geq \dots \geq \frac{x_a^{|K|}}{M_a^{|K|}}$ . The extended formulation then approximates the cost function,  $g_a(x_a^1, x_a^2, \dots, x_a^{|K|})$ , with the function:

$$\hat{g}_a(x_a^1, x_a^2, \dots, x_a^{|K|}) = \sum_{k=1}^{|K|} e_a^k x_a^k, \quad (26)$$

where

$$e_a^l = \frac{g_a(\sum_{k=1}^l M_a^k) - g_a(\sum_{k=1}^{l-1} M_a^k)}{M_a^l}, \quad l = 1, 2, \dots, |K|. \quad (27)$$

We can interpret  $e_a^l$  as the slope of the segment between the cost of the sum of the first  $l - 1$  commodities and the cost of the first  $l$  commodities. For example, in the two-commodity case, if the order of the commodities is 1 then 2, then  $e_a^1$  equals the slope of segment 1 in the dotted line shown in Figure 5 and  $e_a^2$  equals the slope of segment 2. The equation for the lighter of the two triangles in Figure 5 is  $\hat{g}_a(x_a^1, x_a^2) = e_a^1 x_a^1 + e_a^2 x_a^2$ . This result provides another visual interpretation of the improvement gained from the extended formulation: as seen in Figure 5, the approximation provided by the dotted lines improves upon the lower convex envelope provided by the basic formulation.

An extension of the geometric and graphic interpretations discussed for the concave

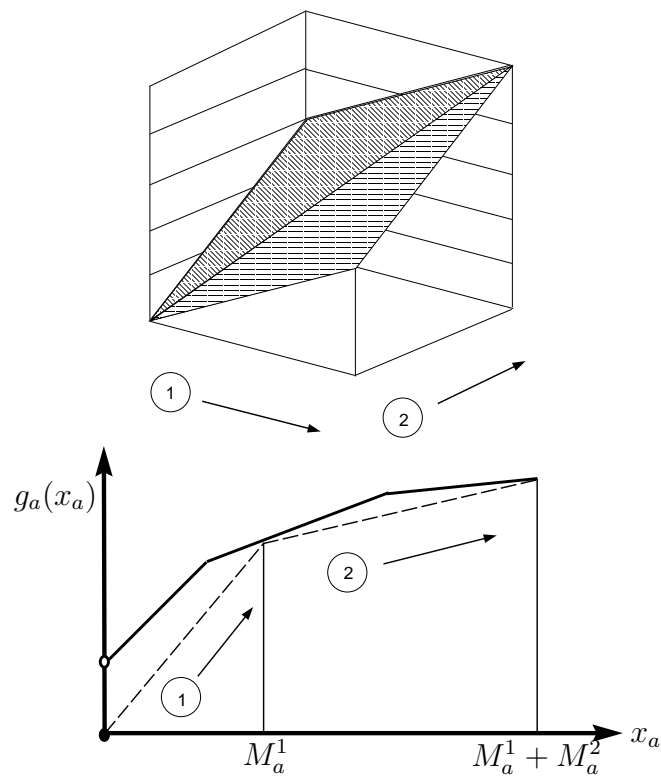


Figure 5: Interpretation of the Cost Coefficients for the Extended Approximation

case applies to non-concave cases, but the closed-form expression given by equation (26) is no longer valid.

## 6 Computational Experiments

In this section, we present computational results on a large set of randomly generated instances, with a variety of cost structures. In all instances, each commodity has a single origin and multiple destinations. Therefore, we can apply classic variable disaggregation techniques that consist of redefining the set of commodities so that each “new” commodity is a pair (“old” commodity, destination). In the resulting reformulation of the problem, each commodity has a single origin and a single destination (the single O-D case). We will compare these two commodity definitions, single O-D and single origin-multiple destinations, within the basic, strong and extended formulations. We are primarily concerned with comparing the LP relaxations of the various formulations. To solve the LP relaxations, we used the dual simplex algorithm of CPLEX (version 7.1). We found that of the algorithms available in CPLEX, the dual simplex usually performed the best on our set of instances.

In addition to learning how the formulations compare with each other, we also wish to determine how tight each of these formulations is as compared with the optimal MIP solution. For small problems, we are able to use the branch-and-bound (B&B) algorithm implemented in CPLEX to solve the MIP to optimality. On larger problems, we used a rounding heuristic to find upper bounds on the optimal value. This heuristic first solves the LP relaxation and examines the optimal values of the binary  $y$  variables. It then fixes to 1 all the  $y$  variables that have values above a user-specified threshold. In our tests, we used a threshold of 0.7. The heuristic also fixes to zero all the  $y$  variables whose value in the optimal LP relaxation solution is zero. With these variables fixed, we then perform B&B on the resulting restricted problem. This restricted problem might not have a feasible solution, but if it does, this solution provides an upper bound. For each of our instances, we always found a feasible solution with this heuristic for at least one of the various formulations we tested, so we could always obtain an upper bound.

We performed the experiments on a Sun Enterprise 10000 with 64 CPUs (with each CPU operating at 450 MHz) and 64 GBs of RAM memory. For each problem instance and for any formulation tested, we set a limit of 1 hour of CPU time for running each of the three phases (LP relaxation, heuristic and B&B) of the algorithm.

To analyze the results, we provide two performance measures for each problem instance and each formulation: (1) the gap,  $\text{GAP}(\%)$ , between the optimal LP objective value, LP, of each formulation and the best known upper bound, UB, computed as  $\text{GAP} = [(\text{UB}-\text{LP}) / \text{LP}] * 100$ ; and (2) the CPU time in seconds, CPU(s), to solve the LP relaxation of each formulation.

## 6.1 Set of Instances

We obtained the problem instances from a network generator similar to the one described in [19, 20] for multi-commodity capacitated fixed-charge problems. When provided with target values for  $|V|$ ,  $|A|$ , and  $|K|$ , this generator creates arcs by connecting two randomly selected nodes (no parallel arcs are allowed). It selects commodities by also choosing at random one origin and 10 destinations for each commodity. We generated the variable costs, capacities, and demands as uniformly distributed over user-provided intervals. We could then scale the capacities to obtain networks with various degrees of capacity tightness, using the *capacity ratio*,  $C = |A|T / \sum_{a \in A} u_a$ , with  $T = \sum_{k \in K} \sum_{i \in O(k)} d_i^k$  the total demand flowing through the network. We adjusted the capacities so that this ratio comes close to user-provided values. As  $C$  approaches 1, the network is lightly capacitated. It becomes more congested as  $C$  increases.

For each network, we generated two cost structures: concave and non-concave. For both types of instances, we provided the maximum number of segments of the cost function,  $S$  as a parameter. For concave instances, we randomly generated a set of decreasing variable costs within the specified interval for each arc. We also set  $b_a^s = \frac{s^2 T}{S^2}$ , for each arc  $a$ , so that the segment length increases as  $s$  increases, as is typical of transportation costs [3]. We then adjusted the number of segments on each arc  $a$  so that  $b_a^{|S_a|} = \min\{T, u_a\}$ , with  $u_a$  as the capacity of the arc. Given variable costs,

breakpoints, and  $f_a^1$ , the initial fixed cost, we can then compute the appropriate fixed costs for the remaining segments so that the resulting function is concave. We obtained non-concave instances by imposing  $b_a^s = \lceil \frac{T}{S} \rceil$ , for each arc  $a$ , so that each segment is of equal size, except the last one. We then adjusted the number of segments to account for the capacities on the arcs. The network generator provided the variable costs, which are not necessarily decreasing, as in the concave case. Given an initial fixed cost  $f_a^1$  for each arc, the remaining fixed costs are computed as  $f_a^s = s f_a^1$ . When  $f_a^1 = 0$ , the resulting function contains no fixed cost on any of the segments, while when  $f_a^1 > 0$ , we obtain a staircase cost function. These functions arise in several applications in production [24], telecommunications [30] and logistics [15].

We classified the generated instances according to seven main characteristics:

1. The cost structure, concave or non-concave.
2. The initial fixed cost: we tested two values, 0 and 1000.
3. The number of commodities: we distinguished between single-commodity and multi-commodity instances, the latter having 10 commodities (recall that each commodity is defined by a single origin and 10 destinations).
4. The network size (NET), represented by the pair  $(|V|, |A|)$ ; we generated networks having the sizes (20,75), (20,100), (25,100) and (25, 150).
5. The maximum number of segments (SEG) of the cost function: we generated instances with a maximum of 4, 6 or 8 segments.
6. The maximum value of each variable cost (VAR): we tested two values, 4 and 10 (we fixed the minimum value of each variable cost to 1).
7. The capacities on the arcs (CAP): we distinguished uncapacitated instances, U, and capacitated instances, C; the first satisfy  $u_a = T$ , while we obtained the second by setting the capacity ratio to the value 2.

We analyzed the results obtained on the resulting 384 generated instances based upon these characteristics.

## 6.2 Performance Analysis on Single-Commodity Instances

Our goal with the computational analysis is to understand the impact of disaggregation on the quality of the LP relaxation for different cost structures. We are able to draw several conclusions by examining the simpler single-commodity case. We compare three formulations: the single origin-multiple destinations basic model; the single origin-single destination strong model, which disaggregates the commodity by destination; and the single origin-single destination extended model, which further disaggregates the flow variables by segment. We will refer to these as the basic, strong, and extended models. Tables 1 and 2 summarize the results obtained on the concave and non-concave instances, respectively (where ext. is an abbreviation for extended). We distinguish instances with no initial fixed cost ( $f_a^1 = 0$ ) and those with an initial fixed cost of 1000 ( $f_a^1 = 1000$ ). The tables isolate four characteristics of the instances, NET, SEG, VAR and CAP, and compare their results, GAP(%) and CPU(s), averaged over all instances having the corresponding value for the given characteristic. Thus, each instance contributes to four entries in each table. For each characteristic, we indicate, between parentheses, the number of instances used to compute the average results.

These results suggest several conclusions that are helpful in choosing when each model is the most appropriate to use. The first is that, as one might expect, the less linear the cost function, the more improvement we gain from disaggregation by adding the extended forcing constraints. This can be seen, for instance, in the concave results, by noting that as the number of segments increases, the gap between the basic and extended formulations increases. This is a result of the method we used to define the concave cost functions, since a function with more segments is “less linear.”

A second set of conclusions can be drawn by examining the effect of the initial fixed cost. As a result of Corollary 3, the strong formulation does not improve upon the basic one when the initial fixed cost is zero (as can be seen in Tables 1 and 2). When the initial



|     |          | $f_a^1 = 0$ |        |      | $f_a^1 = 1000$ |        |      |     |
|-----|----------|-------------|--------|------|----------------|--------|------|-----|
|     |          | Basic       | Strong | Ext. | Basic          | Strong | Ext. |     |
| NET | (20,75)  | 12.7        | 12.7   | 1.6  | 53.5           | 8.3    | 6.3  |     |
|     |          | 0           | 0      | 0    | 0              | 1      | 3    |     |
|     | (12)     | (20,100)    | 10.1   | 10.1 | 2.1            | 58.6   | 5.0  | 3.5 |
|     |          |             | 0      | 0    | 0              | 1      | 4    |     |
|     |          | (25,100)    | 11.1   | 11.1 | 1.1            | 59.6   | 3.8  | 2.7 |
|     |          | 0           | 0      | 0    | 0              | 1      | 4    |     |
|     | (25,150) | 9.2         | 9.2    | 0.3  | 62.7           | 4.0    | 1.7  |     |
|     |          | 0           | 0      | 0    | 0              | 2      | 4    |     |
| SEG | 4        | 7.2         | 7.2    | 0.7  | 57.0           | 4.6    | 3.3  |     |
|     |          | 0           | 0      | 0    | 0              | 1      | 3    |     |
|     | (16)     | 6           | 10.7   | 10.7 | 1.4            | 58.5   | 4.8  | 3.0 |
|     |          |             | 0      | 0    | 0              | 1      | 4    |     |
|     |          | 8           | 14.5   | 14.5 | 1.7            | 60.3   | 6.4  | 4.4 |
|     |          | 0           | 0      | 1    | 0              | 2      | 4    |     |
| VAR | 4        | 8.0         | 8.0    | 0.3  | 64.3           | 4.2    | 3.5  |     |
|     |          | 0           | 0      | 0    | 0              | 1      | 5    |     |
|     | (24)     | 10          | 13.6   | 13.6 | 2.3            | 52.9   | 6.4  | 3.7 |
|     |          | 0           | 0      | 0    | 0              | 1      | 3    |     |
| CAP | U        | 12.1        | 12.1   | 0.8  | 63.0           | 3.1    | 1.2  |     |
|     |          | 0           | 0      | 0    | 0              | 1      | 3    |     |
|     | (24)     | C           | 9.4    | 9.4  | 1.7            | 54.2   | 7.4  | 5.9 |
|     |          | 0           | 0      | 0    | 0              | 1      | 4    |     |

Table 1: Results for 48 Single-Commodity Concave Instances: GAP(%) and CPU(s)

|      |         | $f_a^1 = 0$ |        |      | $f_a^1 = 1000$ |        |      |      |
|------|---------|-------------|--------|------|----------------|--------|------|------|
|      |         | Basic       | Strong | Ext. | Basic          | Strong | Ext. |      |
| NET  | (20,75) | 20.7        | 20.7   | 3.9  | 24.2           | 11.9   | 11.8 |      |
|      |         | 0           | 0      | 0    | 0              | 4      | 8    |      |
|      | (12)    | (20,100)    | 19.6   | 19.6 | 4.5            | 28.5   | 13.4 | 13.4 |
|      |         |             | 0      | 0    | 1              | 0      | 7    | 11   |
|      |         | (25,100)    | 18.8   | 18.8 | 2.2            | 30.4   | 12.7 | 12.5 |
| SEG  |         |             |        |      |                |        |      |      |
|      |         |             | 0      | 0    | 0              | 7      | 15   |      |
|      |         | (25,150)    | 19.3   | 19.3 | 0.9            | 33.1   | 14.1 | 13.8 |
|      |         |             | 0      | 0    | 1              | 0      | 10   | 19   |
|      |         |             |        |      |                |        |      |      |
| (16) | 4       | 15.0        | 15.0   | 2.0  | 37.3           | 9.3    | 9.2  |      |
|      |         | 0           | 0      | 0    | 0              | 4      | 11   |      |
|      | 6       | 20.2        | 20.2   | 2.7  | 28.5           | 15.1   | 14.9 |      |
|      |         | 0           | 0      | 0    | 0              | 7      | 15   |      |
|      | 8       | 23.6        | 23.6   | 3.6  | 21.4           | 14.7   | 14.5 |      |
| VAR  |         |             |        |      |                |        |      |      |
|      |         |             | 0      | 0    | 0              | 9      | 15   |      |
|      |         |             |        |      |                |        |      |      |
|      |         |             |        |      |                |        |      |      |
|      |         |             |        |      |                |        |      |      |
| (24) | 4       | 13.8        | 13.8   | 1.1  | 29.5           | 13.2   | 13.2 |      |
|      |         | 0           | 0      | 0    | 0              | 7      | 14   |      |
|      | 10      | 25.4        | 25.4   | 4.6  | 28.6           | 12.8   | 12.6 |      |
| CAP  |         |             |        |      |                |        |      |      |
|      |         |             | 0      | 0    | 1              | 0      | 7    | 13   |
|      |         |             |        |      |                |        |      |      |
| (24) | U       | 21.6        | 21.6   | 2.1  | 30.0           | 12.4   | 12.3 |      |
|      |         | 0           | 0      | 1    | 0              | 7      | 15   |      |
|      | C       | 17.7        | 17.7   | 3.6  | 28.1           | 13.6   | 13.5 |      |
|      |         | 0           | 0      | 0    | 0              | 6      | 12   |      |

Table 2: Results for 48 Single-Commodity Non-Concave Instances: GAP(%) and CPU(s)

fixed cost is positive, however, the improvement is significant. In addition, the extended formulation improves significantly upon the strong formulation when there is no fixed cost, but only slightly when the fixed costs are large. This is especially the case for non-concave, staircase (i.e. when  $f_a^1 = 1000$ ), cost structures. This result suggests that for large problems with high fixed costs, one might be better to use the strong formulation, as it is only slightly weaker than the extended formulation, but will have fewer variables and constraints. However, for problems with no initial fixed costs, it might be worth the effort of solving the larger extended formulation.

The computational results also show that the basic formulation is particularly weak when the cost structure is concave with a large initial fixed cost, but the strong formulation offers significant improvement. Figure 6 illustrates in more detail the impact of the initial fixed cost for the concave cost case. It shows the gaps for each of the three formulations versus the initial fixed cost, for an uncapacitated instance of size  $|V| = 20$  and  $|A| = 75$ , with concave costs. As we see, the gap for the basic formulation tends to increase as the fixed cost increases. This result is to be expected since the two-dimensional lower convex envelope of the cost function tends to deteriorate as an approximation with an increase in the initial fixed cost, as can also be seen in Figure 3.

The gaps for the two other formulations, on the other hand, show no clear trend. It is interesting to note, however, that when the fixed cost varies from 75 to 150, the gap for the strong formulation decreases, while no trend can be observed for the extended formulation. If we analyze the LP solutions provided by the strong formulation, we can explain this phenomenon. For this network, the flows in the LP optimal solution remain the same for all values of the initial fixed cost between 75 and 150. Similarly, the flows in the MIP optimal solution also remain the same. In addition, for a given fixed cost, both the LP and integer solutions route the flow the same way and the solutions differ only by how the models allocate the flow among the segments. This solution sends the flow for each commodity on a single path. Therefore, in both solutions,  $\sum_{s \in S_a} y_a^s = 1$  for each arc with positive flow. As a result, an increase in the initial fixed cost by  $\Delta$ , increases the optimal cost of both problems by  $\Delta$  on each arc with positive flow. Therefore, when

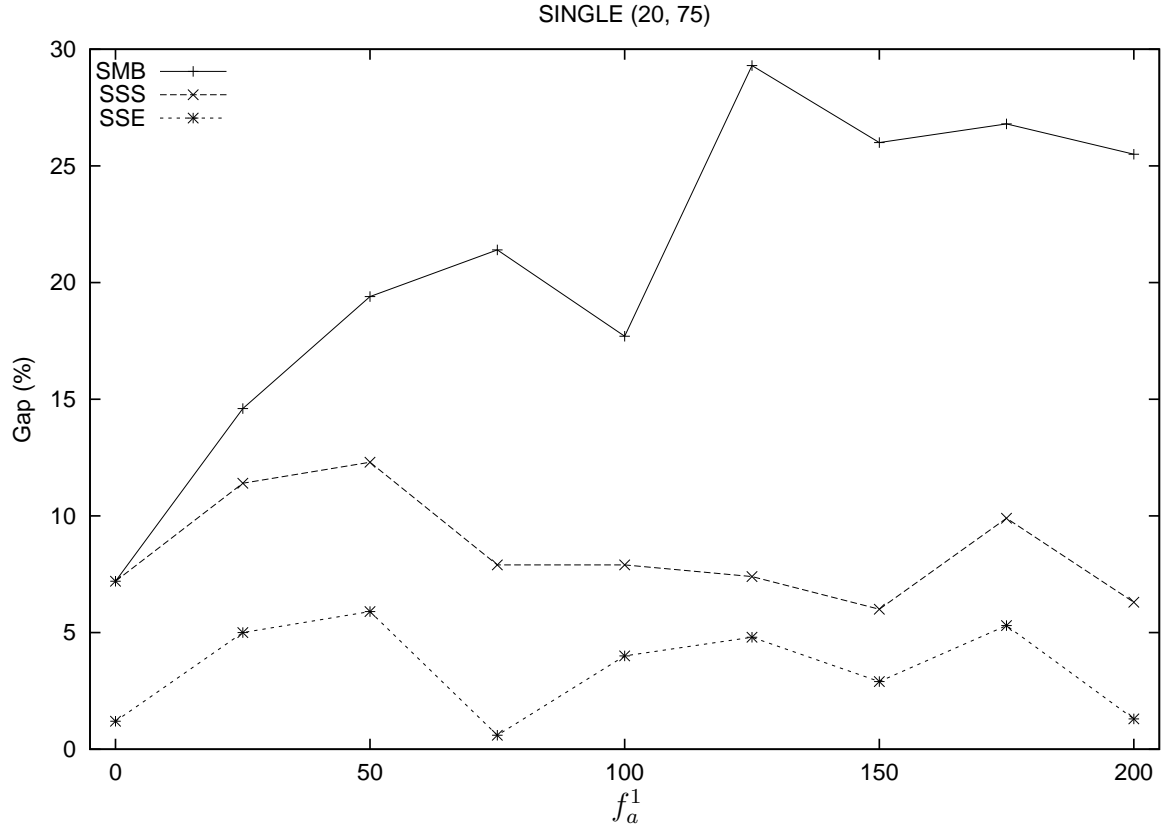


Figure 6: Evolution of GAP Relative to Initial Fixed Cost for a Single-Commodity Instance

we compute the gap using  $[(UB - LP) / LP]$ , the numerator remains constant for all values of the initial fixed cost. The cost of the LP solution, however, increases as the fixed cost increases. Therefore, the relaxation gap for the strong formulation decreases as the fixed cost increases.

This analysis relies on two observations particular to the solutions to this network and demand structure. The first is that the optimal MIP and LP solutions of the strong formulation do not change as the fixed cost changes and, in fact, the routing of the flows is the same in both solutions. We also note that commodities are routed on single paths in this solution.

Although other problems will not always exhibit these two features, they seem to be sufficiently present that the resulting trend often holds; that is, for concave problems, the relaxation gap for the strong formulation tends to decrease as the fixed cost increases.

This result is important because it suggests that problem instances with concave costs and large initial fixed costs are good candidates for using disaggregation techniques.

While the initial fixed cost impacts the relative performance of these formulations, other problem characteristics are also important. For instance, although all formulations are solved quickly, when the initial fixed cost is zero, the problem is more difficult to solve with larger variable costs, especially for non-concave structures. When initial fixed costs are large, however, the impact of the variable costs becomes marginal.

Whether the problem is capacitated also plays a role in the computational results. The basic formulation provides smaller gaps for capacitated instances than uncapacitated ones. This is due to the manner in which the costs were generated because the lower convex envelope in two dimensions provides a better approximation for capacitated instances. For the extended formulation, however, the gaps are larger for capacitated instances, since the capacities tend to cause the flow of each commodity to be split among multiple paths, unlike the uncapacitated instances, for which the solution is often integral (as a result of Proposition 6), or almost integral, especially for concave cost structures.

The number of segments in the cost function can also have an effect. For staircase cost structures (non-concave with  $f_a^1 = 1000$ ), the difference in the gaps between the basic and extended formulations decreases as the number of segments increases, since the basic formulation improves as we increase the number of segments. In addition, the solutions to the strong model and the extended model are often the same, in particular for problems with fewer segments. As the number of segments increases, however, we find instances with a small difference between these two formulations. This result contrasts with the other cost structures for which the gap increases as the number of segments increases. In general, we expect the improvement from using the extended forcing constraints to increase as the number of segments increases, since the strong forcing constraints are aggregated over segments.

And finally, the form of the cost function is important. We find larger gaps for the extended formulation for non-concave costs than for concave costs, but we still observe a remarkable improvement between the basic and extended formulations. Although not

as small as for the concave case, the gaps for the extended formulation are within 5% on average when  $f_a^1 = 0$ . In addition, for staircase cost structures, the LP relaxations are harder to solve than with other cost structures.

### 6.3 Performance Analysis on Multi-Commodity Instances

Further insight can be gained by examining multi-commodity instances. We consider problem instances with 10 commodities, with each commodity still defined with one origin and 10 destinations. Given the weakness of the basic model for single-commodity instances, we focus on formulations that use either the strong or the extended forcing constraints. In defining the commodities, we examine two choices: each commodity is either represented with a single origin and multiple destinations, or we disaggregate them and represent them by a single origin and single destination. We thus compare four different formulations: the single origin-multiple destination strong (MS) model, the single origin-multiple destination extended (ME) model, where we add the extended forcing constraints, the single origin-single destination strong (SS) model, where we disaggregate the commodities by destination and use the strong forcing constraints, and finally, the single origin-single destination extended (SE) formulation, where we disaggregate the variables by destination and use the extended forcing constraints.

We know the fully disaggregated formulation (SE) will provide the tightest bounds, but the resulting problem size might be unwieldy. Therefore, one might want to compromise with one of the intermediate models. Since with these instances we have a choice of disaggregating the commodities by destination or disaggregating the forcing constraints by segment, we are particularly interested in which disaggregation is the most effective under different circumstances. Tables 3 and 4, which are similar to Tables 1 and 2, present our results for the concave and non-concave instances.

Notice that when the problem has no initial fixed cost, the solutions to MS and SS are the same; in other words, disaggregating the commodities has no effect. The explanation for this is similar to the reasoning underlying Corollary 3. From a solution to the LP relaxation of MS, we can derive a feasible solution to the LP relaxation of SS

|     |          | $f_a^1 = 0$ |      |      |      | $f_a^1 = 1000$ |      |      |      |      |
|-----|----------|-------------|------|------|------|----------------|------|------|------|------|
|     |          | MS          | ME   | SS   | SE   | MS             | ME   | SS   | SE   |      |
| NET | (20,75)  | 19.8        | 7.7  | 19.8 | 0.7  | 35.1           | 29.5 | 18.2 | 9.2  |      |
|     |          | 0           | 0    | 4    | 14   | 1              | 1    | 120  | 423  |      |
|     | (12)     | (20,100)    | 15.1 | 7.3  | 15.1 | 0.4            | 43.8 | 40.8 | 23.7 | 19.4 |
|     |          |             | 0    | 0    | 4    | 13             | 2    | 2    | 329  | 1337 |
|     |          | (25,100)    | 17.3 | 7.9  | 17.3 | 0.6            | 50.4 | 46.9 | 29.9 | 24.3 |
|     |          |             | 0    | 0    | 5    | 17             | 1    | 2    | 273  | 1075 |
|     | (25,150) | 9.3         | 4.7  | 9.3  | 0.3  | 55.5           | 53.1 | 33.4 | 30.2 |      |
|     |          | 0           | 0    | 4    | 8    | 2              | 2    | 577  | 2186 |      |
| SEG | 4        | 11.2        | 4.5  | 11.2 | 0.3  | 40.2           | 37.1 | 18.6 | 14.0 |      |
|     |          | 0           | 0    | 2    | 4    | 1              | 1    | 214  | 596  |      |
|     | (16)     | 6           | 16.2 | 7.5  | 16.2 | 0.5            | 46.6 | 42.9 | 26.7 | 21.1 |
|     |          |             | 0    | 0    | 4    | 14             | 1    | 2    | 318  | 1237 |
|     |          | 8           | 18.7 | 8.7  | 18.7 | 0.7            | 51.7 | 47.7 | 33.5 | 27.2 |
|     |          | 0           | 0    | 6    | 21   | 2              | 2    | 441  | 1932 |      |
| VAR | 4        | 11.7        | 5.8  | 11.7 | 0.3  | 48.3           | 46.2 | 27.2 | 24.0 |      |
|     |          | 0           | 0    | 3    | 8    | 2              | 2    | 427  | 1689 |      |
|     | (24)     | 10          | 19.1 | 8.0  | 19.1 | 0.6            | 44.1 | 39.0 | 25.3 | 17.6 |
|     |          | 0           | 0    | 5    | 19   | 1              | 1    | 222  | 822  |      |
| CAP | U        | 13.8        | 6.8  | 13.8 | 0.2  | 50.8           | 47.8 | 28.9 | 24.6 |      |
|     |          | 0           | 0    | 3    | 8    | 2              | 2    | 320  | 1341 |      |
|     | (24)     | C           | 16.9 | 7.0  | 16.9 | 0.7            | 41.6 | 37.4 | 23.7 | 17.0 |
|     |          | 0           | 0    | 5    | 18   | 1              | 1    | 329  | 1170 |      |

Table 3: Results for 48 Multi-Commodity Concave Instances: GAP(%) and CPU(s)

|     |         | $f_a^1 = 0$ |      |      |      | $f_a^1 = 1000$ |      |      |      |      |
|-----|---------|-------------|------|------|------|----------------|------|------|------|------|
|     |         | MS          | ME   | SS   | SE   | MS             | ME   | SS   | SE   |      |
| NET | (20,75) | 23.0        | 11.3 | 23.0 | 2.0  | 38.7           | 38.7 | 18.8 | 18.4 |      |
|     |         | 0           | 0    | 2    | 66   | 2              | 4    | 487  | 1962 |      |
|     | (12)    | (20,100)    | 17.6 | 10.8 | 17.6 | 1.2            | 48.1 | 48.1 | 19.4 | 19.4 |
|     |         |             | 0    | 0    | 2    | 55             | 5    | 7    | 1030 | 2555 |
|     |         | (25,100)    | 20.6 | 11.7 | 20.6 | 2.2            | 48.5 | 48.5 | 16.7 | 16.7 |
|     |         | 0           | 0    | 2    | 62   | 4              | 7    | 937  | 2323 |      |
|     |         | (25,150)    | 12.3 | 6.8  | 12.6 | 0.9            | 55.2 | 55.0 | 18.7 | 18.5 |
|     |         |             | 0    | 0    | 3    | 29             | 10   | 12   | 1743 | 2727 |
| SEG | 4       | 13.3        | 7.8  | 13.3 | 1.1  | 49.3           | 49.2 | 13.4 | 13.0 |      |
|     |         | 0           | 0    | 2    | 11   | 3              | 5    | 434  | 887  |      |
|     | (16)    | 6           | 19.0 | 10.3 | 19.0 | 1.2            | 49.9 | 49.9 | 19.7 | 19.7 |
|     |         |             | 0    | 0    | 2    | 45             | 5    | 9    | 878  | 2734 |
|     |         | 8           | 22.8 | 12.4 | 22.8 | 2.0            | 46.4 | 46.4 | 22.2 | 22.1 |
|     |         | 0           | 1    | 3    | 103  | 8              | 10   | 1835 | 3554 |      |
| VAR | 4       | 13.7        | 8.1  | 13.7 | 0.9  | 48.7           | 48.6 | 19.7 | 19.6 |      |
|     |         | 0           | 0    | 2    | 29   | 7              | 9    | 1342 | 2627 |      |
|     | (24)    | 10          | 23.1 | 12.2 | 23.1 | 2.0            | 48.3 | 48.3 | 17.1 | 17.0 |
|     |         | 0           | 0    | 2    | 77   | 4              | 6    | 756  | 2157 |      |
| CAP | U       | 16.7        | 9.8  | 16.7 | 0.9  | 50.5           | 50.5 | 18.6 | 18.6 |      |
|     |         | 0           | 0    | 2    | 37   | 6              | 9    | 1031 | 2301 |      |
|     | (24)    | C           | 20.0 | 10.6 | 20.0 | 2.0            | 46.5 | 46.4 | 18.2 | 17.9 |
|     |         | 0           | 0    | 2    | 69   | 4              | 6    | 1067 | 2482 |      |

Table 4: Results for 48 Multi-Commodity Non-Concave Instances: GAP(%) and CPU(s)



by increasing  $y_a^1$ . When  $f_a^1 = 0$ , this solution will have the same cost as the solution to MS. This property holds for all cost structures.

Many of our conclusions for the single-commodity instances apply to these multi-commodity instances as well. We note, however, that the gaps shown for capacitated problems with  $f_a^1 = 1000$  are smaller than for uncapacitated instances, contrary to what we observed in the single-commodity case. A careful examination of the results reveals that for several instances, the best upper bound provided by the heuristic is worse for problem instances with no capacities, which induces larger gaps. For instances solved to optimality, however, the gaps for uncapacitated instances are always better than for capacitated ones, which confirms the results obtained with single-commodity networks.

We also see that all the formulations generally exhibit larger gaps when the problem has an initial fixed cost. The SE model, which disaggregates the commodities and the forcing constraints, still improves significantly upon the other formulations, except for staircase cost structures, for which formulation SS is almost equivalent. When the problem has no initial fixed cost, the solution to formulation SE is often integral, especially for uncapacitated concave structures, but we also notice a substantial improvement from just introducing the extended forcing constraints without disaggregating the commodities by destination (i.e., comparing MS and ME). When the problem has an initial fixed cost, we do not obtain any significant improvement without disaggregating the commodities.

We can gain further insight into these formulations by examining the trend in the gaps as we modify the initial fixed cost. In Figure 7, we see how the gaps change as we vary the magnitude of the initial fixed cost for an uncapacitated network of size  $|V| = 20$  and  $|A| = 75$  and with concave costs. As the fixed cost increases, the gaps for MS, ME and SE tend to increase. We see, however, that the gap for formulation SS tends, as we already observed in Figure 6, to decrease. As a result, for small fixed costs, formulation ME is superior to formulation SS, but for large fixed costs, formulation SS is tighter than ME.

We might ask why the gap for formulation MS behaves differently than the gap for formulation SS, since both have the same constraint structure. The rationale goes back

to the discussion of why the relaxation gap for the strong formulation decreases. Recall that this result relies on the single path property. For formulation MS, a commodity is defined by one origin, but multiple destinations. Since the destinations for each origin are likely to be spread throughout the network, we would not expect the flow to multiple destinations to “stick together.” Therefore, the LP solution to MS will not satisfy the single path property. As a result, the relaxation gap for MS does not decrease as the fixed cost increases, as it does for SS.

Figure 7 also provides a view of the trade-off between disaggregating the commodities versus disaggregating the forcing constraints. We see that as the fixed cost increases, the gain from disaggregating the commodities increases (i.e., the difference between MS and SS and the difference between ME and SE both increase), while the value of disaggregating the forcing constraints decreases (i.e., the difference between MS and ME and the difference between SS and SE both decrease). This is an important consideration when choosing a formulation. In determining which formulation is the most appropriate, it is clear that we should consider the relative magnitude of the fixed costs. These results suggest that the fully disaggregated formulation (SE) is most effective for problems with concave costs and with small initial fixed costs. If, however, we want to use a smaller formulation, we should keep the commodities aggregated and use formulation ME whenever the problem has small initial fixed costs. When the problem has larger initial fixed costs, disaggregating the commodities but using the strong forcing constraints (formulation SS) might be the most appropriate.

In Figure 8, we vary the number of destinations that each origin serves, for the same network as in the previous figure, but with an initial fixed cost of 1000. In this case, we retain the same expected supply out of each origin. Therefore, as we increase the number of destinations served, we decrease the expected demand for each origin-destination pair. We see that, with an increasing number of destinations per origin, the gaps for the formulation with aggregated commodities (MS and ME) tend to increase, while the gap for SS tends to decrease. This effect is similar to the one observed in the previous figure. With an increasing number of destinations per origin, the demand

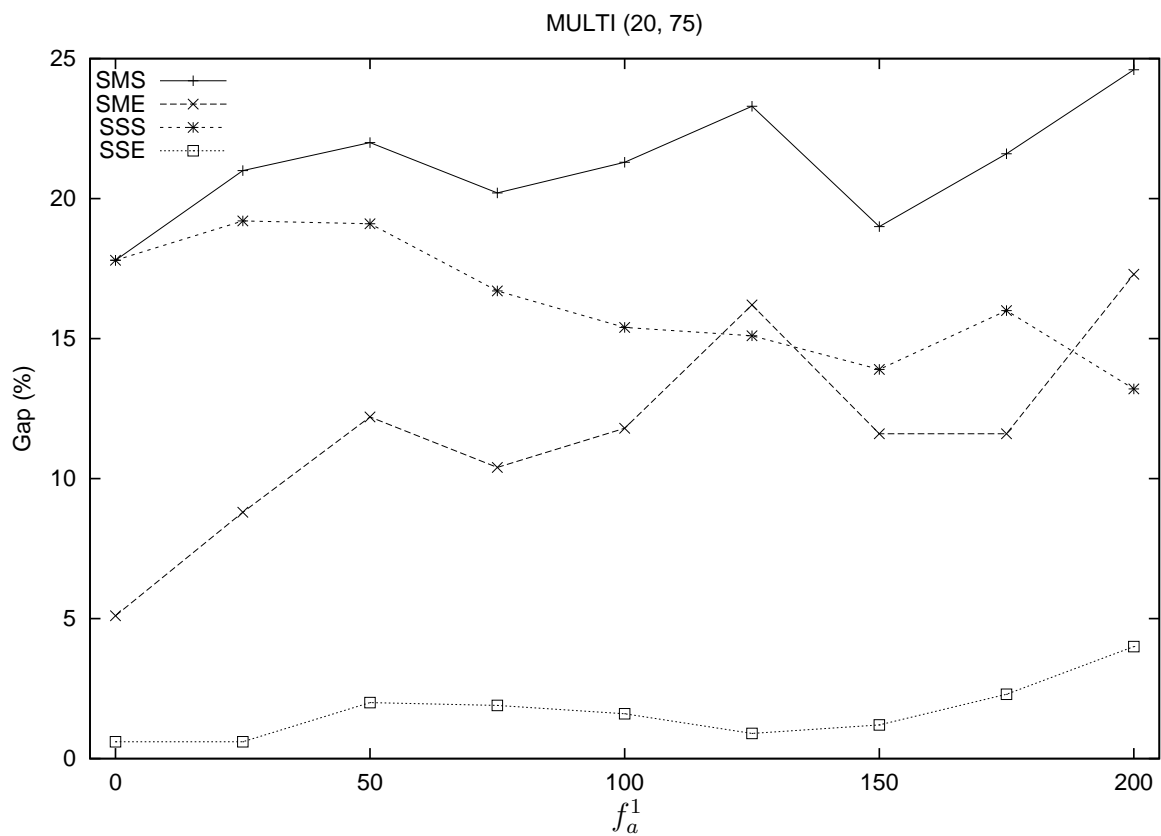


Figure 7: Evolution of GAP Relative to Initial Fixed cost for a Multi-Commodity Instance

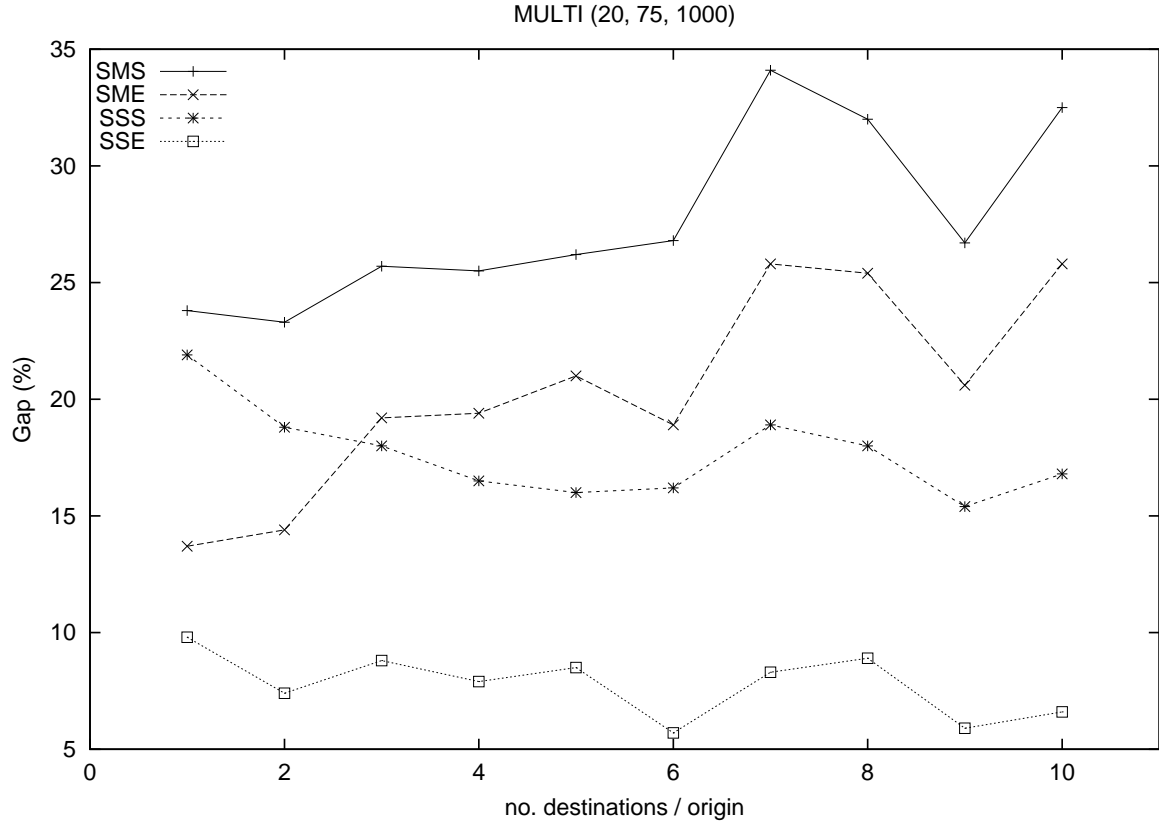


Figure 8: Evolution of GAP Relative to Number of Origins per Destination for a Multi-Commodity Instance

of the individual commodities decreases. Therefore, the relative magnitude of the fixed costs to the demand of each commodity increases. Also, as expected, the difference between MS and SS and the difference between ME and SE both increase. This outcome reflects the intuitive notion that as the number of commodities increase, the gains from disaggregating by commodity also increase.

## 7 Conclusions

We studied three MIP formulations for modeling multi-commodity network flow problems with piecewise linear cost functions: basic, strong and extended formulations. We first provided an upper bound on the improvement in the objective value that the strong LP relaxation could provide over the basic LP relaxation. We recalled a result of Croxton, Gendron and Magnanti [14], who showed that the LP relaxation of the basic formulation

estimates the cost function with its lower convex envelope, where the cost is seen as a function of the total flow on each arc. We generalized this result by showing that the extended formulation approximates the cost function with its lower convex envelope in the space of commodity flows, the cost on each arc being represented as a function of the  $|K|$  variables corresponding to the flow for each commodity. For the extended LP relaxation, we also derived a characterization of LP integral solutions. With these results, we examined a two-commodity example and developed a geometrical view of the impact of disaggregation on a single arc.

Through computational experiments we studied how this impact plays out over an entire network. The computational results reported here allowed us to compare the solutions of different LP relaxations, as well as to a computed upper bound. These results showed that disaggregation techniques can significantly improve LP lower bounds for multi-commodity network flow problems with piecewise linear costs. In addition, they suggested the following guidelines when choosing among the formulations:

- The less linear the cost function, the more the expected improvement from disaggregation.
- The relative magnitude of the initial fixed costs plays an important role. For instance:
  - The basic formulation is particularly weak when the cost structure is concave with a large initial fixed cost. These problem instances are good candidates for using disaggregation.
  - For cost functions with no initial fixed cost, the strong formulation does not improve upon the basic formulation (Corollary 3).
  - For cost functions with large initial fixed costs, the strong formulation improves significantly over the basic formulation, and the extended formulation provides only slight additional improvement.
  - For concave cost functions, the relaxation gap for the strong formulation tends to decrease as the fixed cost increases.

- On multi-commodity problems, if the size of the formulation is an issue and it is necessary to choose between disaggregating the commodities or disaggregating the forcing constraints, it is preferable to disaggregate the forcing constraints when fixed costs are small, but to disaggregate the commodities when fixed costs are high.
- The basic formulation performs better for capacitated problems than uncapacitated ones, but the gaps for the extended formulation are larger for capacitated problems.
- The improvement gained from using the extended formulation increases as the number of segments in the cost function increase.
- The extended formulation is not as tight on instances with staircase costs as it is for other cost structures.
- On a multi-commodity network, the benefit from disaggregating by commodity increases as the number of destinations served by each commodity increases.

These results provide computational evidence that disaggregation can be quite powerful when applied to multi-commodity network flow problems with general piecewise linear costs. However, for staircase cost functions and large networks, the LP relaxations are computationally difficult to solve, so the trade-off between solution time and formulation tightness requires close examination. Solving large problems in practice often requires more sophisticated approaches. For example, Croxton, Gendron and Magnanti [15] exploit the disaggregation techniques discussed in the present paper to develop a cutting-plane algorithm for the merge-in-transit problem, a large-scale multi-commodity network flow problem with piecewise linear costs.

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## Appendix: Proof of the Lower Convex Envelope Property by Extreme Point Characterization

We provide an alternative proof of the lower convex envelope property (Theorem 5), based on a characterization of the extreme points of the polyhedron associated with the LP relaxation of the extended formulation. More specifically, we will show that for any arc  $a$  with flow values of  $\widehat{x}_a^k$  for each commodity  $k$ , the objective value of the LP relaxation of the extended formulation obtained by optimally choosing the other variables is given by the lower convex envelope of the cost function on arc  $a$ . Since the lower convex envelope of a separable sum of functions (defined over a bounded polyhedron) equals the sum of the lower convex envelopes of these functions [16], the lower convex envelope property follows directly. We prove this by first characterizing the extreme points of the polyhedron associated the LP relaxation of the extended formulation. We then show that the minimum cost convex combination of these extreme points defines the lower convex envelope of the cost function on each arc when it is represented in  $|K| + 1$  dimensions.

As in the proof of the lower convex envelope property by Lagrangian duality, let  $P_a$  be the polyhedron associated to the LP relaxation of the extended model on a single arc,  $a$ , with no constraints on the flow of each commodity (i.e.,  $P_a = \{(x_a, y_a) : b_a^{s-1} y_a^s \leq \sum_{k \in K} x_a^{ks} \leq b_a^s y_a^s, \forall s \in S_a; x_a^{ks} \leq M_a^k y_a^s, \forall s \in S_a, k \in K; \sum_{s \in S_a} y_a^s \leq 1; y_a^s \geq 0, \forall s \in S_a; x_a^{ks} \geq 0, \forall s \in S_a, k \in K\}$ ). The same proof shows that every extreme point of  $P_a$  satisfies  $y_a^s \in \{0, 1\}, \forall s \in S_a$ , implying that for each arc  $a$  with positive flow, every extreme point of  $P_a$  satisfies the condition  $y_a^t = 1$ , for some  $t \in S_a$ , and  $y_a^s = 0, s \neq t$ . We can write the corresponding polyhedron associated with segment  $t$  as:  $P_a^t = \{x_a : b_a^{t-1} \leq \sum_{k \in K} x_a^{kt} \leq b_a^t; x_a^{kt} \leq M_a^k, x_a^{kt} \geq 0, \forall k \in K\}$ . Any extreme point of  $P_a^t$  must satisfy at least  $|K|$  linearly independent constraints as an equality. Only two kinds of points will satisfy this requirement. One is the set of points with  $x_a^{kt} \in \{0, M_a^k\}, \forall k \in K$ . We refer to these points as *corner points* because they correspond to the corners of the feasible  $|K|$ -dimensional hypercube (thus, among these corner points, we include the origin, which is the unique extreme point of  $P_a$  corresponding to the situation when an arc has zero

flow). The other is the set of points with  $\sum_{k \in K} x_a^{kt} = b_a^t$  or  $b_a^{t-1}$ , and  $x_a^{kt} \in \{0, M_a^k\}$  for at least  $|K| - 1$  of the  $|K|$  variables. We refer to these as *ridge points* and they correspond to the extreme points of the  $|K|$ -dimensional face defined by  $\sum_{k \in K} x_a^{kt} = b_a^t$  (or  $b_a^{t-1}$ ) and  $0 \leq x_a^{kt} \leq M_a^k$ . Note that, if we represent the cost function,  $g_a(x_a^1, x_a^2, \dots, x_a^{|K|})$ , in  $|K| + 1$  dimensions, these corner and ridge points correspond to the extreme points of the  $|K| + 1$ -dimensional segments of the cost function.

Now, consider  $\widehat{P}_a$ , the polyhedron associated with the LP relaxation of the extended formulation on a single arc with flow values of  $\widehat{x}_a^k$  for each commodity  $k$ :  $\widehat{P}_a = \{(x_a, y_a) \in P_a : \sum_{s \in S_a} x_a^{ks} = \widehat{x}_a^k, \forall k \in K\}$ . Since  $\widehat{P}_a$  is obtained from  $P_a$  by adding  $|K|$  linear equalities, it is easy to show that any extreme point of  $\widehat{P}_a$  is a convex combination of at most  $|K| + 1$  of the extreme points of  $P_a$ , i.e., the corner and ridge points corresponding to the extreme points of the  $|K| + 1$ -dimensional segments of the cost function. Therefore, as we vary the values of  $\widehat{x}_a^k$ , the cost minimizing convex combination will be given by the lower convex envelope of the cost function,  $g_a(x_a^1, x_a^2, \dots, x_a^{|K|})$ .